Combinants, Bell polynomials and applications

1984 J. Phys. A: Math. Gen. 17 989

(http://iopscience.iop.org/0305-4470/17/5/022)

J. Phys. A: Math. Gen. 17 (1984) 989–1002. Printed in Great Britain

Combinants, Bell polynomials and applications

R Vasudevan†, P R Vittal‡ and K V Parthasarathy‡

† MATSCIENCE, The Institute of Mathematical Sciences, Madras-600 113, India
‡ Department of Mathematics, Vivekananda College, Madras-600 004, India

Received 28 June 1983

Abstract. The concept of combinants introduced in the formulation of the generating function for probabilities is analysed, demonstrating the fact that they play the same role in computing cumulants as probabilities do in computing moments. The mathematical framework of Bell polynomials is used to relate combinants and probabilities. The effective use of combinants in branching processes is brought out. Also the coupled differential equations governing the combinants yield direct coupled equations for cumulants. The concept of mixed combinants is developed. This will be explored in later contributions.

1. Introduction

Study of point processes is very important in many areas in physics, engineering, biology and a host of other fields of human activity, and the realisations of such a point process are point events.

Such problems are analysed by formulation of certain point functions such as cumulant functions (Kendall 1949) and product densities (Ramakrishnan 1950) etc. The product densities (§ 2) formulated by Ramakrishnan provide a complete characterisation of the process and also satisfy elegant equations. Kauffman and Gyulassy (1978), studying theoretical models for created boson multiplicities, introduced quantities called 'combinants' which measure the deviation of a given process from those described by Poisson probabilities (§ 3). The purpose of this paper is to study the relationship of combinants with the well known correlation and cluster functions, cumulants etc (§ 4). Bell polynomials (Riordan 1958) provide the essential mathematical framework relating combinants, probabilities etc (§ 5). Applications of this concept to complex branching processes, the analysis of emitted photoelectrons and other phenomena are dealt with in detail (§ 6).

2. Product densities

The product densities, a powerful tool to deal with stochastic point processes, relate to the distribution of a discrete number of entities in a continuous infinity of states (Ramakrishnan 1959). The central quantity of interest is $\mathrm{d}N(t)$, the number of entities occurring in a continuous interval $(t, t+\mathrm{d}t)$. Assuming that the probability that there is one entity between t and $t+\mathrm{d}t$ is of order $\mathrm{d}t$ and the probability that there is more than one is less than $\mathrm{d}t$, we get the mean number in $\mathrm{d}t$ as

$$E[\mathrm{d}N(t)] = f_1(t)\,\mathrm{d}t \qquad (2.1)$$

where $f_1(t)$ is called the product density of first order.

Product densities of higher orders express all the correlations of the stochastic variable $dN(t)$ existing at various t values. A very useful result for the calculation of the rth moment of the number of entities in the desired range runs as

$$E[(N(b)-N(a))^r]=\sum_{s=1}^{r} C_s^r \int_a^b dt_1 \int_a^b dt_2 \dots \int_a^b dt_s f_s(t_1, t_2, \dots, t_s) \tag{2.2}$$

where C_s^r denotes the number of various confluences of the infinitesimal intervals, the maximum order of any confluence being $(r-s)$. A closed form for C_s^r is (Kuznetsov *et al* 1965, Vasudevan 1969)

$$C_s^r=\frac{1}{s!}\left(\frac{d^r}{d\omega^r}(e^\omega-1)^s\right)_{\omega=0}=\frac{1}{s!}\sum_{k=0}^{s}\binom{s}{k}k^r(-1)^{s-k}. \tag{2.3}$$

The higher-order product densities $fr(x_1, x_2, \dots, x_r)$ can be expressed in terms of cluster functions $g_r(x_1, x_2, \dots, x_r)$ which are not separable. The g's are irreducible cluster functions which characterise the internal correlation in the system which are related to connected diagrams in other phraseologies. For a Poisson process all g's other than g_1 are zero.

The moment generating function of a discrete process governed by probabilities $P(n)$ can be related to the product density generating function as

$$\sum_{n=0}^{\infty} P(n)\, e^{\omega n}=\sum_{s=0}^{\infty}\frac{(e^\omega-1)^s}{s!}\int_R \dots \int_R f_s(x_1, x_2, \dots, x_s)\, dx_1 \dots dx_s = Q_m(\omega). \tag{2.4}$$

Inversely, the probability $P_R(n)$ of n entities in region R is given by

$$P_R(n)=\frac{1}{n!}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\int_R \dots \int_R f_{n+s}(x_1, x_2, \dots, x_{n+s})\, dx_1 \dots dx_{n+s}. \tag{2.5}$$

The factorial moment generating function $Q_m(\omega')$ is given by replacing ω by $\log(1+\omega')$ in (2.4). Also,

$$\left(\frac{\partial^r}{\partial \omega'^r}[Q_m|\omega']\right)_{\omega'=0}=\left\langle\frac{n!}{(n-r)!}\right\rangle=\int_R \dots \int_R f_r(x_1, x_2, \dots, x_r)\, dx_1 \dots dx_r. \tag{2.6}$$

The cumulant generating function is given by

$$Q_c(\omega)=\log Q_m(\omega)=\sum_{s=1}^{\infty}\frac{(e^\omega-1)^s}{s!}\tau_s$$

where

$$\tau_s=\int_R \dots \int_R g_s(x_1, \dots, x_s)\, dx_1 \dots dx_s \tag{2.7}$$

and g_s are the cluster functions of order s relating to the point process given rise to by $P(n)$.

The cumulants K_r of the $P(n)$ process are obtained from $Q_c(\omega)$ by

$$K_r=[\partial^r Q_c(\omega)/\partial \omega^r]_{\omega=0}. \tag{2.8}$$

Hence we easily see that

$$K_r=\sum_{s=1}^{r} C_s^r \tau_s \tag{2.9}$$

where C_s^r are the same coefficients as in (2.3) (Kuznetsov 1965, Ziff 1977).

To obtain the factorial cumulant function we can replace in (2.7) the parameter ω by $\log(1+\omega')$. We then see that τ_s' are the factorial cumulants, similar to the factorial moments given by (2.6). What is the signficance of the cumulants and factorial cumulants? The moments and the factorial moments are derived from sums over the probabilities. If so, what is the analogous situation in the case of cumulants? We will examine these ideas in the context of the entities called the combinants to be described in § 3.

3. Combinants

The probability generating function for the Poisson distribution with mean $\bar{n}$ is given by

$$F(\lambda)=\sum_{n=0}^{\infty}\lambda^n P(n)=\exp[(\lambda-1)\bar{n}]. \tag{3.1}$$

We see that $\log F(\lambda)$ is a first degree polynomial in λ. If however, we employ a higher degree polynomial, we can write in general

$$\log F(\lambda)=\log P(0)+\sum_{k=1}^{\infty} c(k)\lambda^k \qquad \text{with } P(0)>0 \tag{3.2}$$

which means

$$F(\lambda)=\exp\left(\sum_{k=1}^{\infty} c(k)(\lambda^k-1)\right) \qquad \text{and} \qquad \exp-\sum_1^{\infty} c(k)=P(0). \tag{3.3}$$

The coefficients $c(1), c(2), \ldots$ thus completely characterise the probabilities $P(n)$ and express the deviations from a simple Poisson process. The $c(k)$'s are expressible in terms of the first k probability ratios $P(1)/P(0), P(2)/P(0), \ldots, P(k)/P(0)$. It should, however, be noted that the condition $P(0)>0$ is necessary for the existence of the $c(k)$ defined as above.

The expressions for $c(k)$'s in terms of $(P(k)/P(0))$'s are given as follows. For example $c(3)$ is given as

$$c(3)=\frac{P(3)}{P(0)}-\left(\frac{P(1)}{P(0)}\right)\left(\frac{P(2)}{P(0)}\right)+\frac{1}{3}\left(\frac{P(1)}{P(0)}\right)^3 \tag{3.4}$$

and the general expression for any k is obtained in § 4 using Bell polynomials.

Inversely, $P(n)$'s are given in terms of $c(k)$'s as

$$P(n)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\left(\prod_{k=1}^{\infty}\frac{(c(k))^{n_k}\,e^{-c(k)}}{n_k!}\right)\delta\left(n,\sum_{r=1}^{\infty} rn_r\right) \tag{3.5}$$

which is derived by us using the mathematical framework of Bell polynomials in § 4.

Now, in calculating cumulants, combinants play the same role as the probabilities $P(n)$ do in computing moments. We display

$$\sum_{l=1}^{\infty} l^r c(l)=K_r \qquad \text{just as the } r\text{th moment} \qquad \sum_{n=0}^{\infty} n^r P(n)=\mu_r. \tag{3.6}$$

To see this let us start from the moment generating function

$$Q_{\mathrm{m}}(\omega) = P(0)\exp\left(\sum_{k=1}^{\infty} c(k)\,\mathrm{e}^{\omega k}\right)$$
$$= \sum_{s=0}^{\infty} \frac{(\mathrm{e}^{\omega}-1)^s}{s!}\int_R \ldots \int_R f_s(x_1, x_2, \ldots, x_s)\,\mathrm{d}x_1 \ldots \mathrm{d}x_s$$
$$= \exp \sum_{s=1}^{\infty} \frac{(\mathrm{e}^{\omega}-1)^s}{s!}\tau_s \tag{3.7}$$

using earlier equations.

Taking logarithms on both sides in (3.7), we get

$$\log P(0) + \sum_{k=1}^{\infty} c(k)\,\mathrm{e}^{\omega k} = \sum_{s=1}^{\infty} \frac{(\mathrm{e}^{\omega}-1)^s}{s!}\tau_s. \tag{3.8}$$

Differentiating both sides, with respect to ω, r times and giving the value $\omega = 0$, we get

$$\sum_{k=1}^{\infty} k^r c(k) = K_r = \sum_{s=1}^{r} C_s^r \tau_s. \tag{3.9}$$

Thus, we see that the rth cumulants are obtained taking the average $\langle l^r \rangle$ over all the combinants $c(l)$ and cumulants in turn are related to the cluster integrals τ_s by C_s^r coefficients in a way analogous to the moment relationships with the integrals over the product densities. Taking $\omega = \log(1+\omega')$ and differentiating both sides suitably and putting $\omega' = 0$, we get

$$\sum_l \frac{l!}{(l-r)!} c(l) = \tau_r. \tag{3.10}$$

Thus τ_r is the rth factorial cumulant with respect to $c(l)$. This is analogous to the factorial moments as displayed in (2.6.).

It is also easily seen that

$$c(l) = \frac{1}{l!}\sum_{s=1}^{\infty} \frac{(-1)^s}{s!}\tau_{s+l}. \tag{3.11}$$

Thus we have seen that the combinants play very much the same role as the probabilities themselves. We compute cumulants and factorial cumulants with respect to $c(l)$ in the same manner as we compute moments and factorial moments with respect to $P(n)$.

4. Bell polynomials and combinants

Bell polynomials, which play a big role in analysing partition functions in terms of cluster integrals (Mayer and Mayer 1977), are introduced here to arrive at the relations between combinants and probabilities, detailed in the earlier sections. For composite functions of the type $F(t) = f[g(t)]$, assuming $f(0) = g(0) = 0$, we obtain (Aldrovandi and Monte Lima 1980) in general

$$F(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} F_n(f; g) \tag{4.1}$$

where

$$F_n = \sum_{l=1}^{n} f_l B_{nl}(g_1, \ldots, g_{n-l+1}), \tag{4.2}$$

with f_r and g_r being the coefficients occurring in the Taylor expansion of $f(t)$ and $g(t)$ respectively. $B_{nl}(g)$ are called the Bell polynomials connected with the function g and are defined by

$$B_{nl}(g) = (1/l!)[\mathrm{d}^n(g(t))^l/\mathrm{d}t^n]_{t=0} \tag{4.3}$$

and can be easily expressed in an explicit form using the multinomial theorem (Abramowitz and Stegun 1965).

The terms $B_{nl}(g)$ can be considered as the elements of the left-triangular infinite matrix $B_n(g)$ (if n becomes infinite) and the group property of such matrices leads to useful relations such as

$$B_{jl}[f(g(t)] = \sum_{n=l}^{j} B_{jn}[g(t)]B_{nl}[f(t)]. \tag{4.4}$$

Taking $g(t) = \mathrm{e}^t - 1$, we see that

$$B_{nl}[g(t)] = B_{nl}(1, 1, \ldots, 1) = \frac{1}{l!}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\mathrm{e}^t - 1)^l\right]_{t=0} = S_l^n \tag{4.5}$$

and this is exactly the same as the C_l^n coefficients obtained in (2.3). These are also known as the Stirling numbers of the second kind. Stirling numbers of the first kind are obtained by the inverse of the matrix $B_{nl}(g)$. This means we sould take the function f as $\log(1+u)$ and obtain its $B_{nl}(f)$. They are

$$B_{nl}(f) = B_{nl}(0!, -1!, 2!, -3!, \ldots). \tag{4.6}$$

In a similar fashion, we can find the $P(n)$'s by the composite function formula, if we take

$$\tilde{F} = [F(\lambda)/P(0) - 1] \tag{4.7}$$

so that

$$\tilde{F}(0) = 0,$$

$$g(\lambda) = \sum_{k=1}^{\infty} c(k)\lambda^k, \qquad g(0) = 0, \tag{4.8}$$

$$f(u) = \mathrm{e}^u - 1, \qquad f(0) = 0. \tag{4.9}$$

Hence, we easily see that

$$n!\frac{P(n)}{P(0)} = \tilde{F}_n(\lambda) = \sum_{l=1}^{n} B_{nl}[1!c(1), 2!c(2), \ldots, (n-l+1)!c(n-l+1)]. \tag{4.10}$$

Therefore the probability ratios $P(n)/P(0)$ are given in terms of $c(k)$'s. This is exactly the formula (3.5), when we recognise that $\exp - \Sigma_k\, c(k) = P(0)$ and that when l exceeds n all the B_{nl} quantities are zero. To obtain the combinants $c(k)$ themselves, we could use the matricial expression for the composite function $\tilde{F}(\lambda)$, namely

$$B[\tilde{F}(\lambda)] = B[f(g(t))] = B[g(t)]B[f(u)]. \tag{4.11}$$

Hence

$$B_{n1}(g) = g_n = n!c(n) = \sum_{l=1}^{n} B_{nl}[\tilde{F}]B_{l1}^{-1}(f). \tag{4.12}$$

The $B_{l1}^{-1}(f)$ elements of the inverse function of f can be obtained as in (4.6) and hence we have

$$c(n) = \frac{1}{n!}\sum_{l=1}^{n} B_{nl}\left(\frac{P(1)}{P(0)}, \frac{P(2)}{P(0)}2!, \frac{P(3)}{P(0)}3!, \ldots\right)(l-1)!(-1)^{l-1}. \tag{4.13}$$

and $B_{nl}[\tilde{F}(\lambda)]$ are given by

$$\frac{1}{l!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\left(\sum_k \lambda^k \frac{P(k)}{P(0)}\right)^l\right]_{\lambda=0}.$$

Hence we have the values of the first few $c(k)$ as

$$c(1) = \frac{P(1)}{P(0)}, \qquad c(2) = \frac{P(2)}{P(0)} - \frac{1}{2}\left(\frac{P(1)}{P(0)}\right)^2,$$

$$c(3) = \frac{P(3)}{P(0)} - \left(\frac{P(1)}{P(0)}\right)\left(\frac{P(2)}{P(0)}\right) + \frac{1}{3}\left(\frac{P(1)}{P(0)}\right)^3 \tag{4.14}$$

etc as in (3.4).

5. Combinants and branching process

The basic mechanism of a branching process (Harris 1963) is to start with an initial individual of the zeroth generation who is capable of producing progeny with probabilities $P(k)$, $k = 0, 1, 2, \ldots$ etc who constitute the first generation. Each individual of this generation produces offspring independent of each other with probabilities which are the same for the individuals in each generation. If this probability of reproduction remains constant, the generating function for the population of the Nth generation is

$$F_N(\lambda) = F_1[F_{N-1}(\lambda)] = F_{N-1}[F_1(\lambda)]. \tag{5.1}$$

For details, see Bailey (1964). Here we employ the concept of combinants to get the results for cumulants etc for the Nth generation.

The moment generating function ${}^N Q_{\mathrm{m}}$ of the Nth generation is

$${}^N Q_{\mathrm{m}}(\omega) = \exp\left(\sum_{k=1}^{\infty} {}^N c(k)(\mathrm{e}^{k\omega} - 1)\right) \tag{5.2}$$

where ${}^N c(k)$ are the combinants corresponding to the Nth generation. From (5.1), we can see that

$${}^N Q_{\mathrm{m}} = \exp\left(\sum_{k=1}^{\infty} {}^1 c(k)(\exp k\,{}^{N-1}Q_{\mathrm{c}} - 1)\right) \tag{5.3}$$

where

$${}^{N-1}Q_{\mathrm{c}} = \left(\sum_{l=1}^{\infty} {}^{N-1}c(l)(\mathrm{e}^{l\omega} - 1)\right) \tag{5.4}$$

and ${}^{1}c(k)$ are the combinants of the first generation. Hence, let us take the cumulant generating function of the Nth generation as

$$^{N}Q_{c}=\sum_{k=1}^{\infty}{}^{1}c(k)[\exp(k\,^{N-1}Q_{c})-1]. \tag{5.5}$$

We see that the recursion for the cumulant generating function ${}^{N}Q_{c}$ in terms of ${}^{N-1}Q_{c}$ etc can be written for all $N=1,2,3,\ldots$. We thus obtain

$$^{N}Q_{c}(\omega)={}^{1}Q_{c}[{}^{N-1}Q_{c}(\omega)]={}^{1}Q_{c}\{{}^{1}Q_{c}\{{}^{N-2}Q_{c}(\omega)\}\}. \tag{5.6}$$

We easily see that ${}^{N}Q_{c}(0)=0$ for all N and thus for these composite functions, the techniques of the Bell polynomials can be effectively used to obtain the recursion relations for ${}^{N}k_{r}$, the rth cumulant of the Nth generation. The Bell polynomial matrix corresponding to the sth generation is given by the elements ${}^{(s)}B_{nl}$ by using (4.3)

$$^{(s)}B_{nl}\{{}^{(s)}Q_{c}(\omega)\}=\left[\frac{1}{l!}\left(\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\{{}^{(s)}Q_{c}\}^{l}\right)\right]_{\omega=0}. \tag{5.7}$$

Here ${}^{(N)}B_{r1}={}^{(N)}K_{r}$ the rth cumulant of the Nth generation. Equation (5.6) can be easily generalised as

$$^{N}Q_{c}={}^{(1)}Q_{c}({}^{(1)}Q_{c}({}^{(1)}Q_{c}\ldots)). \tag{5.8}$$

Therefore the ${}^{(N)}B_{nl}$ polynomial matrix is nothing but the ${}^{(1)}\mathrm{B}_{\mathrm{nl}}$ matrix multiplied by itself N times. That is

$$^{(N)}B_{nl}=[{}^{(1)}B_{nl}]^{N}$$

$$=\begin{bmatrix} {}^{(1)}K_{1} & 0 & 0 & \ldots & 0\\ {}^{(1)}K_{2} & {}^{(1)}K_{1}^{2} & 0 & \ldots & 0\\ {}^{(1)}K_{3} & 3{}^{(1)}K_{1}^{(1)}K_{2} & {}^{(1)}K_{1}^{3} & \ldots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ {}^{(1)}K_{n} & . & . & & {}^{(1)}K_{1}^{n}\end{bmatrix}^{N}. \tag{5.9}$$

The ${}^{(1)}K_{r}$ denotes the rth cumulant for the first generation. It is easy to check the usual recursion formula for the Nth and $(N-1)$th generations (Bailey 1964).

Since B_{nl} is a lower diagonal matrix the $[B_{nl}]^{N}$ matrix will have its diagonal elements ${}^{(1)}K_{1}^{N}, {}^{(1)}K_{1}^{2N}, {}^{(1)}K_{1}^{3N}, \ldots, {}^{(1)}K_{1}^{N^{2}}$. Hence if ${}^{(1)}\mathrm{K}_{1}$, which is the mean of the first generation, is less than unity, the diagonal elements will tend to zero as N tends to ∞ which signifies extinction.

If all the branching processes are of the same Poisson type with mean m

$$^{1}Q_{c}(\omega)=m(\mathrm{e}^{\omega}-1) \tag{5.10}$$

then the B_{nl} matrix corresponding to this function is

$$\begin{bmatrix} S_{1}^{1}m & 0 & 0 & \ldots & 0\\ S_{1}^{2}m & S_{2}^{2}m^{2} & 0 & \ldots & 0\\ S_{1}^{3}m & S_{2}^{3}m^{2} & S_{3}^{3}m^{3} & \ldots & 0\\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix}={}^{1}B_{nl}. \tag{5.11}$$

The matrix of ${}^{(N)}B_{nl}$ coefficients corresponding to the Nth generation is given by the Nth power of this matrix. We obtain the first column of the second generation matrix as

$$^{2}B_{n1}=[m^{2}, m(m+m^{2}), m(m+S_{2}^{3}m^{2}+S_{3}^{3}m^{3}),\ldots]\qquad \text{for } n=1,2,\ldots. \tag{5.12}$$

From the cumulant generating function of the second generation, we have the combinants of the second generation as

$$^2c(r) = m(e^{-m}m^r/r!) \qquad \forall r = 1, 2, \ldots . \tag{5.13}$$

Hence by equating cumulants of the second generation, we obtain the identity

$$\sum_{r=1}^{\infty} \frac{r^n e^{-m} m^r}{r!} = m + S_2^n m^2 + \ldots + S_n^n m^n = \sum_{l=1}^{n} m^l S_l^n. \tag{5.14}$$

This should be true for all values of m. This identity can also be arrived at by other methods. We can also express the combinants of the rth generation in terms of the combinants of the $(r-1)$th generation. We obtain

$$^rc(l) = \left(\sum_k e^{-km} k^{l\, r-1} c(k)\right) \frac{m^l}{l!}. \tag{5.15}$$

It is also easy to see that the probability of zero population at the rth generation is given by the formula

$$\log {^r p(0)} = {^{r-1}Q_c}(\log p(0)) \tag{5.16}$$

where $^rp(0)$ is the probability of zero population of the rth generation and $^{r-1}Q_c$ is the cumulant generating function of the $(r-1)$th generation. The above equation can also be written as

$$\log {^r p(0)} = Q_c(\log^{r-1} p(0)). \tag{5.17}$$

Even if the production probabilities are not Poisson the above equation is true. Hence, if $\log p(i) = \theta_0$ for any generation, $\theta_0 = Q_c(\theta_0)$ is the fixed point equation that has to be solved for computing the extinction probability for any population (see Bailey 1964).

To calculate the second cumulant of the Nth generation, it is enough if we concentrate on the 2×2 B_{nl} matrix of the form

$$B_{nl} = \begin{bmatrix} K_1 & 0 \\ K_2 & K_1^2 \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & K_1^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ K_2 & 0 \end{bmatrix} = D + M. \tag{5.18}$$

We should raise this 2×2 matrix to the power N. It is important to note that the lower diagonal matrix M raised to the power $N \geq 2$ will vanish. Hence if we write the B_{nl} matrix as a sum of the diagonal and lower diagonal matrix as in (5.18), when we compute $(D+M)^N$, the only non-vanishing terms will be of the type

$$D^N + \sum_{l=0}^{N-1} D^l M D^{N-l-1} = [B_{nl}]^N = \begin{bmatrix} K_1^N & 0 \\ 0 & K_1^{2N} \end{bmatrix} + \sum_{l=0}^{N-1} \begin{bmatrix} 0 & 0 \\ K_1^{2l} K_2 \quad K_1^{N-l} & 0 \end{bmatrix}. \tag{5.19}$$

Hence, the second cumulant of the Nth generation is given by

$$^N K_2 = \sum_{l=0}^{N-1} K_1^{2l} K_2 K_1^{N-l-1}. \tag{5.20}$$

Since we know that $K_2 = \sigma^2$ and $K_1 = m$ for the production probabilities, the K_2 of the Nth generation is

$$\sigma^2[(1-m^N)/(1-m)]m^{N-1}. \tag{5.21}$$

To compute the third cumulant of the Nth generation, it is enough if we take the

3×3 B_{nl} matrix given by

$$B_{nl}=\begin{bmatrix}K_1 & 0 & 0\\ K_2 & K_1^2 & 0\\ K_3 & 3K_1K_2 & K_1^3\end{bmatrix}=\begin{bmatrix}K_1 & 0 & 0\\ 0 & K_1^2 & 0\\ 0 & 0 & K_1^3\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ K_2 & 0 & 0\\ 0 & 3K_1K_2 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ K_3 & 0 & 0\end{bmatrix}$$

$$=D+M_1+M_2. \tag{5.22}$$

After raising the above matrix to the power N, we are only concerned with the ${}^{N}B_{31}$th term of the Nth generation matrix. To this end it is enough if we compute the sum of the following type of products of the matrices:

$$\sum_{r=0}^{N-r}\sum_{l=0}^{N-1} D^l M_1 D^r M_1 D^{N-l-r-2}+\sum_{l=0}^{N-1} D^l M_2 D^{N-l-1}. \tag{5.23}$$

The term corresponding to the third cumulant of the Nth generation is obtained by considering the Nth power of the 3×3 B_{nl} matrix. This can be obtained from (5.23) as

$$N_{K_3}=\frac{K_1^{N-1}(1-K_1^N)}{1-K_1^2}\left(\frac{3K_2^2(1-K_1^{N-1})}{1-K_1}+K_3(1+K_1^N)\right). \tag{5.24}$$

We can easily check this out for the basic Poisson production process, whose cumulants are all m. Calculations in the Poisson case become simpler if we take note of the fact that

$$DM_1=mM_1D \qquad \text{and} \qquad DM_2=mM_2D.$$

6. Combinants and other applications

6.1. Birth, death and immigration process

In treating a linear birth and death process with individual probabilities being time independent or otherwise, using the concept of combinants, we can directly write down the equations for the cumulants in a compact form, without solving for the generating function. As usual, we have the inflow–outflow equations for $P(n,t)$, the probability for having n units in time t as

$$\mathrm{d}P(n,t)/\mathrm{d}t=-(\lambda+\mu)nP(n,t)+\mu(n+1)P(n+1,t)+\lambda(n-1)P(n-1,t) \tag{6.1}$$

with $P(n,0)=\delta_{n,1}$.

The partial differential equation for $G(u,t)$, the probability generating function with parameter u, is

$$\partial G(u,t)/\partial t=(1-u)(\mu-\lambda u)\partial G/\partial u. \tag{6.2}$$

Taking the expression $G(u,t)=\exp[\Sigma_{k=1}^{\infty} c(k,t)(u^k-1)]$, we obtain the equation satisfied by combinants $c(k,t)$ as

$$\mathrm{d}c(k,t)/\mathrm{d}t=\lambda[(k-1)c(k-1,t)-kc(k,t)]$$
$$-\mu[kc(k,t)-(k+1)c(k+1),t)], \qquad k=1,2,\ldots, \tag{6.3}$$

Multiplying both sides by k^r and summing, we obtain the equation for the rth cumulant

K_r as

$$\frac{dK_r}{dt} = \lambda \sum_{l=0}^{r-1} \binom{r}{l} K_{l+1} + \mu \sum_{l=0}^{r-1} (-1)^{r-l} \binom{r}{l} K_{l+1},$$
$$K_r(0) = \delta_{r,1}. \tag{6.4}$$

Hence for the first moment $K_1 = m$ we have

$$dm(t)/dt = (\lambda - \mu)m \qquad \text{with } m(0) = 1 \tag{6.5}$$

giving

$$m(t) = e^{(\lambda-\mu)t} \tag{6.6}$$

and

$$d\sigma^2(t)/dt = (\lambda + \mu)m(t) + 2(\lambda - \mu)\sigma^2, \qquad \text{with } \sigma(0) = 0 \tag{6.7}$$

with the solution

$$\sigma(t) = [(\lambda + \mu)/(\lambda - \mu)](e^{3(\lambda-\mu)t} - e^{2(\lambda-\mu)t}). \tag{6.8}$$

The actual solution obtained for $G(u, t)$ from (6.2)

$$G(u, t) = \frac{\mu(1 - e^{-(\lambda-\mu)t}) - (\mu - \lambda\, e^{-(\lambda-\mu)t})u}{(\lambda - \mu\, e^{-(\lambda-\mu)t}) - \lambda(1 - e^{-(\lambda-\mu)t})u}. \tag{6.9}$$

The solution of (6.3), for the combinants $c(k, t)$, can be found as $c(k, t) = (\alpha^k - \beta^k)/k$, with

$$\alpha = \frac{\lambda(1 - e^{-(\lambda-\mu)t})}{\lambda - \mu\, e^{-(\lambda-\mu)t}}, \qquad \beta = \frac{\mu - \lambda\, e^{-(\lambda-\mu)t}}{\mu(1 - e^{-(\lambda-\mu)t})}. \tag{6.10}$$

If besides the linear birth and death process, we have addition of a unit to the population with probability $\nu\, dt$ between time t and $t + dt$ due to immigration, (6.1) will be modified as

$$dP(n, t)/dt = -(\lambda + \mu)nP(n, t) + \mu(n+1)P(n+1, t)$$
$$+ \lambda(n-1)P(n-1, t) + \nu P(n-1, t) - \nu P(n, t). \tag{6.11}$$

This leads to a system of coupled equations for the $c_I(k, t)$ combinants for the process with immigration as

$$\frac{dc_I(k, t)}{dt} = \lambda[(k-1)c_I(k-1, t) - kc_I(k, t)]$$
$$- \mu[kc_I(k, t) - (k+1)c_I(k+1, t)] + \nu c_I(k, t)\delta_{k,1}. \tag{6.12}$$

In the limit t tending to ∞, we can arrive at the stationary state $c_I(k)$ obtained by summing the coupled set (6.12) as

$$c_I(k) = (\nu/\mu)(\lambda/\mu)^{k-1} k^{-1}. \tag{6.13}$$

In the case $\lambda/\mu < 1$, the cumulants are given by

$$K_r = \frac{\nu}{\mu} \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{k-1} k^{r-1}, \qquad r = 1, 2, 3, \ldots, \tag{6.14}$$

Let us now find the solution for $c_I(k, t)$ at any time t. The immigration of a person at a given time t is independent of the process started by the individual at time $t=0$. Hence, the full generating function for the process is given by

$$G_{\mathrm{I}}(u,t)=G(u,t)\exp\nu\int_0^t[G(u,t-\tau)-1]\,\mathrm{d}\tau. \tag{6.15}$$

Hence in the differential equation for $G_{\mathrm{I}}(u,t)$, we have an additional term $(u-1)G_{\mathrm{I}}$ which again gives the extra term $\nu c_{\mathrm{I}}(k,t)\delta_{k,1}$ in the differential equation for the combinant $c_{\mathrm{I}}(k,t)$. We can easily obtain the solution of (6.12) for finite time as

$$C_{\mathrm{I}}(k,t)=c(k,t)+\nu\int_0^t\sum_{l=1}^{k}B_{kl}(1!c(1),2!c(2),\ldots,(k-l+1)!c(k-l+1))(t-\tau)\,\mathrm{d}\tau \tag{6.16}$$

6.2. Compound or filtered Poisson process

The occurrence of point events may be Poisson with intensity ν and, associated with each event, we can have a random variable whose statistical features have to be studied from the moments of the compound process $Y(t)$ given by

$$Y(t)=\sum_{i=1}^{N(t)}\xi_i,\qquad N(t)\geqslant 1 \tag{6.17}$$

where $N(t)$ is a Poisson process. The generating function of $Y(t)$ is given by (Snyder 1975)

$$G_{Y(t)}(u)=E[\mathrm{e}^{uY(t)}]=\exp\{\nu t[\phi_\xi(u)-1]\} \tag{6.18}$$

where $\phi_\xi(u)$ is the generating function of the ξ process.

We now observe from the composite function

$$\bar{F}(u)=f[g(u)]=G_{Y(t)}(u)-1$$

where

$$g(u)=\phi_\xi(u)-1\qquad\text{and}\qquad f=\mathrm{e}^{\nu t u}-1 \tag{6.19}$$

that

$$E(Y^n)=B_{n1}[\bar{F}(u)]$$

and

$$E(\xi^n)=B_{n1}[g(u)]. \tag{6.20}$$

As in (4.4), the B_{nl} matrix of the composite function $f[g(u)]$ is the product of the B_{nl} matrices corresponding to g and f. To obtain the $B_{nl}(g)$ corresponding to the ξ process, we can find the inverse Bell polynomial matrix $B^{-1}(f(u))$, and use that to find the moments of ξ. Since $[\log(1+u)]/\nu t$ is the inverse of $\mathrm{e}^{\nu ut}-1$, we get after computation, using (4.12),

$$B_{n1}[g(u)]=\frac{1}{\nu t}\sum_{l=1}^{n}B_{nl}[\bar{F}(u)](-1)^{l-1}(l-1)!. \tag{6.21}$$

$B_{nl}[\bar{F}(u)]$ can be easily expressed in terms of the moments of the observed final process. Hence, the statistics of the ξ process can be computed easily.

6.3. Doubly stochastic processes

If we are interested in a simple Poisson process whose Poisson parameter is again a random variable, the generating function for the final process is given by

$$Q_{\mathrm{m}}^{\mathrm{final}} = \langle \exp[w(\mathrm{e}^{\theta} - 1)] \rangle_W \tag{6.22}$$

where W is the Poisson random process. Hence the final Q_{m} should be averaged over this process. Since all the cumulants of the discrete Poisson process are equal to the average $\bar{W}$, we have by Kubo's theorem (Kubo 1962),

$$Q_{\mathrm{m}}^{\mathrm{final}} = \exp \sum_{n=1}^{\infty} \frac{\bar{W}(\mathrm{e}^{\omega} - 1)^n}{n!} = \exp \bar{W}[\exp(\mathrm{e}^{\omega} - 1) - 1]. \tag{6.23}$$

This means that the $c(k)$'s of the final process are

$$c(k)^{\mathrm{final}} = \bar{W} \mathrm{e}^{-1}/k!. \tag{6.24}$$

Since all the $c(k)$'s exist the final process is not Poisson but a correlated process. If however, W is not a discrete process, but a continuous stochastic variable, such as the intensity of the gaussian light falling as a sensitive solid, and if we are interested in the stochastic features of the emitted electrons, we compute the average (Vasudevan 1969)

$$\left\langle \exp\left(\lambda\alpha \int_0^T I(t')\,\mathrm{d}t' \right) \right\rangle_I$$

$$= \exp \sum \frac{\lambda^n \alpha^n}{n!} \int_0^t \int_0^T \dots \int_0^T g_n(t_1, t_2, \dots t_n)\,\mathrm{d}t_1 \dots \mathrm{d}t_n = L(\lambda) \tag{6.25}$$

where g_n are the nth cluster functions and in the case of gaussian light they are the coherence functions. If it is assumed as in Saleh (1978) that the nth-order coherence functions are constant, $(n-1)!\bar{I}^n$, then (6.25) reduces to $\exp[-\log(1 - \alpha\bar{I}T\lambda)]$, which is the factorial moment generating function. Replacing λ by $\lambda - 1$, we get the probability generating function as

$$\exp \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\alpha\bar{I}T}{1 + \alpha\bar{I}T} \right)^n (\lambda^n - 1). \tag{6.26}$$

Hence the combinants $c(k)$ of this process relating to the emission of the electrons are given by

$$c(k) = \frac{1}{k} \left(\frac{\alpha\bar{I}T}{1 + \alpha\bar{I}T} \right)^k = \frac{1}{k} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^k \tag{6.27}$$

where $\bar{n}$ is the average number of the emitted electrons for the gaussian light. Other types of mixing of gaussian light with signals of laser light can be easily obtained and the bunching phenomena can also be studied.

The product density generating function and the probability generating function are related in a simple fashion

$$L(\lambda) = F(\lambda + 1)$$

$$= \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} \int \dots \int f_s(x_1 \dots x_s)\,\mathrm{d}x_1 \dots \mathrm{d}x_s$$

$$= \exp \sum c(k, t)[(\lambda + 1)^k - 1]. \tag{6.28}$$

Using the Bell polynomial method, we can express (6.25) as

$$L(\lambda)=\exp\sum_{k=1}^{\infty}\bar{c}(k,t)\{\lambda^k-1\}. \tag{6.29}$$

Here

$$\bar{c}(l,t)=\{K_l\}/l! \tag{6.30}$$

where $\{K_l\}$ is the lth factorial cumulant τ_l of (2.7). It is well known that equation (6.28) can be expressed as exponential sums of τ_l's with proper coefficients. If the final process represented by $p(n, t)$ is the resultant of different causative phenomena, such as due to mixing of different beams, we know that the sums of cluster integrals of different types should occur in the exponential. Also a cluster function of mixed type will appear in the exponential sum. Hence, if chaotic light is mixed with a laser signal the photoelectron emission can be described by the product density generating function as

$$L(\lambda)=\exp\left(\sum_{n=1}^{\infty}\frac{\lambda^n\bar{I}_c^n}{n}+\lambda\bar{I}_s+\sum_{n=1}^{\infty}\frac{\lambda^{n+1}\bar{I}_s\bar{I}_s^n}{n}\right) \tag{6.31}$$

where $\bar{I}_c^n$ represent the cluster integrals due to gaussian light, $\bar{I}_s$ the only possible cluster integral of the Poisson signal and $\bar{I}_s\bar{I}_c^n$ are the $(n+1)$th cluster integrals due to mixing of the two beams. Hence by putting $\lambda=\lambda'-1$ we can obtain the $P(n, t)$ generating function in terms of combinants as

$$L(\lambda'-1)=F(\lambda').$$
$$=\exp\sum c^{\mathrm{I}}(k,t)(\lambda'^k-1)+\sum c^{\mathrm{II}}(k,t)(\lambda'^k-1)+\sum c^{\mathrm{I,II}}(k,t)(\lambda'^k-1) \tag{6.32}$$

where $c^{\mathrm{I}}(k, t)$ are the combinants due to gaussian light which can be computed from equation (6.32) as

$$c^{\mathrm{I}}(k,t)=k^{-1}[\bar{I}_c/(1+\bar{I}_c)]^k, \tag{6.33}$$

$$c^{\mathrm{II}}(k,t)=\bar{I}_s\delta_{k,1} \tag{6.34}$$

(since this is only a Poisson process) and

$$c^{\mathrm{I,II}}(k,t)=\bar{I}_s[\bar{I}_c/(1+\bar{I}_c)]^k k^{-1} \tag{6.35}$$

due to the mixing part of the beam. Thus we have extended the concept of combinants to entities called mixed combinants. Mixed product densities are common features in dealing with the cascade shower problems as in Ramakrishnan *et al* (1965).

In conclusion we want to point out that the main results of the paper hinge on the fact that combinants play the same role in computing cumulants as probabilities for finding the moments. Also the connection between the integrals of cluster functions and combinants has been brought out. The Bell polynomial analysis of composite functions is utilised to obtain several relationships between combinants and probabilities. Recursion and other types of formulae for cumulants governing branching processes and inverse determination of the statistics of the jumps in a compound Poisson process from the observed final features are also facilitated by the use of Bell polynomial matrices. It is demonstrated that inflow–outflow equations for combinants in the context of birth, death and immigration processes lead directly to the evaluation of cumulants. Finally, the concept of mixed combinants has been introduced and will be exploited in further contributions.

References

Abramowitz M and Stegun A S 1965 *Handbook of Mathematical Functions* (New York: Dover)

Aldrovandi R and Monte Lima I 1980 *J. Phys. A: Math. Gen.* **13** 3685

Bailey N T J 1964 *The Elements of Stochastic Processes* (New York: Wiley)

Harris T E 1963 *The Theory of Branching Processes* (Berlin: Springer)

Kauffmann S K and Gyulassy M 1978 *J. Phys. A. Math. Gen.* **11** 1715

Kendall D G 1949 *J.R. Statist. Soc. B* **11** 230

Kubo R 1962 *J. Phys. Soc., Japan* **17** 1100

Kuznetsov P I, Stratonovich R L and Tikhonov V I 1965 *Nonlinear transformations of Stochastic Processes* (Oxford: Pergamon)

Mayer J E and Mayer J E 1977 *Statistical Mechanics* 2nd edn (New York: Wiley)

Ramakrishnan A 1950 *Proc. Camb. Phil. Soc.* **46** 595

—— 1959 *Probability and Stochastic Processes, Handbuch der Physik*, vol III (Berlin: Springer) p 524

Ramakrishnan A, Vasudevan R and Srinivasan S K 1965 *J. Math. Anal. Applic.* **11** 278

Riordan J 1958 *An Introduction to Combinatorial Analysis* (New York: Wiley)

Saleh B 1978 *Photo-electron Statistics* (Berlin: Springer)

Snyder D L 1975 *Random Point Process* (New York: Wiley)

Vasudevan R 1969 *Photon Statistics and Coherence in light beams, Symp. Theor. Phys. and Math.* vol 9 (New York: Plenum) p 89

Ziff R M 1977 *J. Math. Phys.* **18** 1825